Briefing Notes—Engine Optimization

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IDEA 1: *The Air-Standard Otto Cycle—Basic Theory*

Overview. The Otto cycle is one way to transform heat energy into mechanical work. The following very rough illustration will help us describe it.

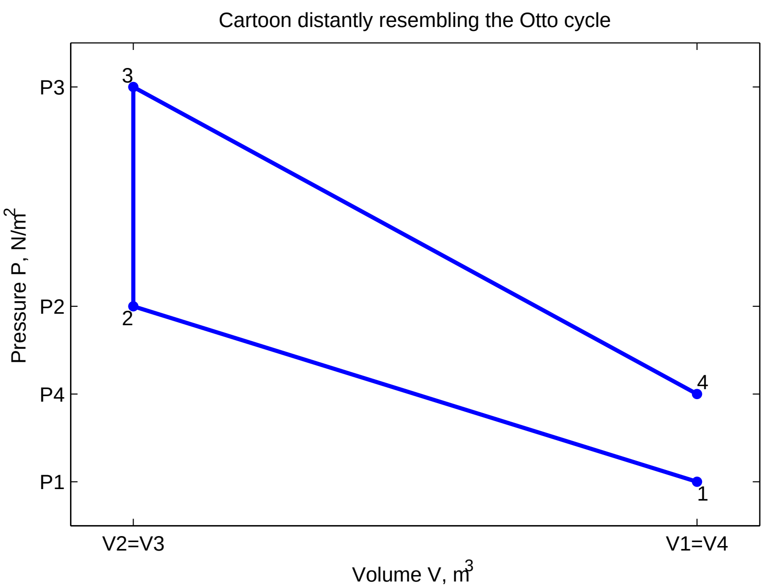


Fig. 1: The Otto Cycle

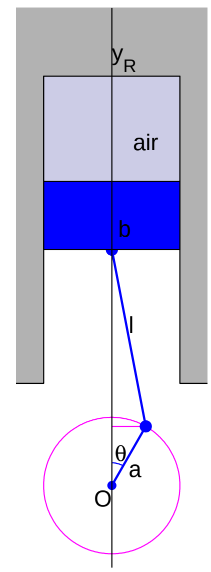


Fig. 2: Piston Geometry

Figure 1 suggests how pressure and volume evolve during the power stroke in one cylinder of a simplified internal-combustion engine. The cylinder is sketched in Fig. 2. In state 1, $\theta = -\pi$ and the piston is at "Bottom Dead Centre (BDC)"—the point where it leaves the maximum volume in the cylinder. (Call that volume V_1.) The cylinder is full of a fuel-air mixture. The piston in the cylinder compresses the gas leading to state 2. This is the compression stroke. In state 2, $\theta = 0$ and the piston is at "Top Dead Centre (TDC)"—the point where the volume in the cylinder (V_2) is smallest. Of course the corresponding pressure P_2 is larger than the initial pressure P_1. Then the spark plug fires and a rapid chemical reaction transforms energy stored in chemical bonds into kinetic energy of the gas molecules. The state jumps to position 3, where V_3 is the same as V_2 (the piston is still at TDC), but P_3 is much higher than P_2. Then the high pressure pushes on the piston as it moves back out to its original position (BDC), now described with crank angle $\theta = \pi$. This is the power stroke. It brings the system to state 4, where the enclosed volume V_4 is the same as the initial volume V_1. Of course the pressure in the cylinder, P_4, is higher than the initial pressure P_1 because the gas has higher temperature. To get back to state 1, ready to repeat the cycle, real engines make a whole extra revolution to expel the combustion products from the cylinder and replace them with a fresh fuel-air mixture. You can find a more detailed description on Wikipedia (`en.wikipedia.org/wiki/Four-stroke_engine`), and a nice animated diagram at `http://www.animatedengines.com/otto.shtml`.

Air-Standard Assumption and Other Idealizations. This lab deals only with the compression and power strokes, so the transition sequence $1 \to 2 \to 3 \to 4$ highlighted in Fig. 1 will be our exclusive focus. Combustion is too complicated to deal with in detail, so we assume the cylinder actually contains nothing but air ($k = 1.4$), and model the transition $2 \to 3$ as the instantaneous addition of some known amount of heat energy to the air by some unspecified process. *These simplifications characterize the "air-standard" model.* To get started, we will assume further that the cycle starts at standard atmospheric pressure $P_1 = 101$ kPa and temperature $T_1 = 333$ K. The mass of air in the cylinder, say m, stays the same throughout the process, so the following pressure-volume-temperature relationship is always in force:

$$PV = mRT. \tag{1}$$

If there is no heat transfer between the gas and the cylinder walls or the piston, the compression ($1 \to 2$) and expansion ($3 \to 4$) processes can be taken as adiabatic and reversible.

Thermodynamics. Two fundamental equations and one definition govern this process. The equations are the ideal gas law (1) and the First Law of Thermodynamics (simplified):

$$\delta Q = P\,dV + mC_v\,dT. \tag{2}$$

The definition is $k = 1 + R/C_v$, where R is the gas constant and C_v is the specific heat of the gas we are using. This leads to a convenient form of the First Law where C_v does not appear explicitly:

$$\delta Q = P\,dV + \frac{mR}{k-1}\,dT, \qquad \text{i.e.,} \qquad dT = \frac{k-1}{mR}\left[\delta Q - P\,dV\right]. \tag{3}$$

In our scenario, the mass m in the cylinder is constant. Thus the gas law (1) implies

$$\frac{dP}{P} + \frac{dV}{V} = \frac{dT}{T}.$$

We can replace the right side here using the First Law in form (3):

$$\frac{dP}{P} + \frac{dV}{V} = \frac{k-1}{mRT}\left[\delta Q - P\,dV\right]. \tag{4}$$

Here the combination mRT on the right side equals PV by (1), so we have the following important relationship between pressure, volume, and heat content:

$$\boxed{\frac{dP}{P} = \frac{k-1}{PV}\,\delta Q - \frac{k}{V}\,dV.} \tag{5}$$

In an adiabatic, reversible process, $\delta Q = 0$ and (5) reduces to a differential equation describing a famous relationship between P and V:

$$\frac{1}{P}\frac{dP}{dV} = -\frac{k}{V}. \tag{6}$$

In Fig. 1, the transition from state 1 to state 2 should follow a path compatible with (6); likewise for the transition from state 3 to state 4. The true paths are *not straight lines*!

In any reversible piston-moving scenario, the mechanical work done by the gas on the outside world is

$$W = \int dW = \int P\,dV. \tag{7}$$

This is a "line integral", so orientation matters. In the transition from state 1 to state 2, $dV < 0$ leads to a negative result: this makes sense because it's the compression stroke, in which the gas is receiving energy from outside. By contrast, in the transition from state 3 to state 4, $dV > 0$ and the contribution to W is positive because the gas is expanding and sending mechanical energy to the outside. (In the idealized case of Fig. 1, where state 2 and state 3 have the same volume, we have $dV = 0$ for the segment joining them. Since the piston doesn't move, no work goes either way.)

The efficiency of heat-to-work conversion along the path $1 \to 2 \to 3 \to 4$ is the ratio of work output to heat input:

$$\eta = \frac{W}{Q_{\text{in}}},$$

where W is the work in line (7) and $\overline{Q} = \int \delta Q$ is the line integral accounting for heat addition. Taken together, we can express efficiency as a ratio of line integrals:

$$\eta = \frac{W}{Q_{\text{in}}} = \frac{\displaystyle\int P\,dV}{\displaystyle\int \delta Q}. \tag{8}$$

IDEA 2: *Crank Angle Parametrizes Everything*

As the piston makes one cycle through compression, spark, and expansion, the crankshaft at O keeps turning. Thus every point on the state-transition path $1 \to 2 \to 3 \to 4$ in Figure 1 is associated with a specific crank angle θ in the mechanism shown in Figure 2. We can use this variable to parametrize all the paths we need to analyze.

The volume in the cylinder as a function of crank angle θ can be found using standard geometric reasoning (see the Appendix): it is

$$\frac{V(\theta)}{V_d} = \frac{1}{r-1} + \frac{1}{2}\left[1 + R_1 - \cos\theta - \sqrt{R_1^2 - \sin^2\theta}\right]. \tag{9}$$

Here V_d is the piston's displacement, R_1 is a dimensionless ratio of link lengths in the piston mechanism, and r is the dimensionless ratio of maximum to minimum volume in the cylinder. Differentiating with respect to θ gives

$$\frac{V'(\theta)}{V_d} = \frac{1}{2}\sin\theta\left[1 + \frac{\cos\theta}{\sqrt{R_1^2 - \sin^2\theta}}\right]. \tag{10}$$

Angle $\theta = 0$ gives the minimum gas volume, $V(0) = V_d/(r-1)$.

Heat addition is *fast*, but it can't be truly *instantaneous*. So the constant-volume transition between states 2 and 3 shown in Figure 1 is one of the model's more aggressive simplifications. Experts suggest that heat generated by combustion might actually depend on crank angle like this:

$$Q(\theta) = \int_0^\theta \delta Q = \begin{cases} 0, & \text{if } -\pi \le \theta \le \theta_s, \\ \dfrac{\overline{Q}}{2}\left(1 - \cos\left(\dfrac{\pi(\theta - \theta_s)}{\theta_b}\right)\right), & \text{if } \theta_s < \theta < \theta_s + \theta_b, \\ \overline{Q}, & \text{if } \theta_s + \theta_b \le \theta \le \pi. \end{cases} \tag{11}$$

Here we can see the constant value $Q = 0$ before the "spark angle" θ_s and the higher constant value $Q = \overline{Q}$ for all angles after $\theta_s + \theta_b$. Here θ_b is the "burn angle"—the amount the crank turns during heat addition; it is typically quite small, but not zero.

These features are also visible in the derivative formula

$$Q'(\theta) = \begin{cases} 0, & \text{if } -\pi < \theta \le \theta_s, \\ \dfrac{\pi \overline{Q}}{2\theta_b} \sin\left(\dfrac{\pi(\theta-\theta_s)}{\theta_b}\right), & \text{if } \theta_s < \theta < \theta_s + \theta_b, \\ 0, & \text{if } \theta_s + \theta_b < \theta < \pi. \end{cases} \tag{12}$$

Here are figures showing the functions Q and Q' for certain values of θ_s and θ_b; we take $\overline{Q} = 1800$ J.

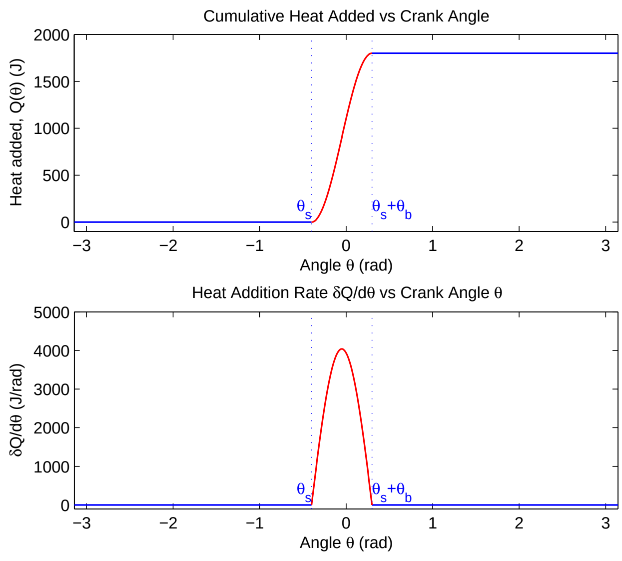


Fig. 3: Heat input rate and total heat added, versus crank angle

The spark angle θ_s is an adjustable feature of most standard internal-combustion engines; the burn angle θ_b can be influenced by chemical properties of the fuel-air mixture and by mechanical considerations outside the scope of our work here. These two angles are important design parameters for engine-makers.

Dividing the boxed equation (5) above by $d\theta$ produces a differential equation that reveals the θ-dependence of the pressure $P = P(\theta)$:

$$\boxed{\frac{dP}{d\theta} = \frac{k-1}{V(\theta)} Q'(\theta) - \frac{kP(\theta)}{V(\theta)} V'(\theta).} \tag{13}$$

We already have detailed formulas for $V(\theta)$, $V'(\theta)$, and $Q'(\theta)$, and we know the initial condition $P(-\pi) = P_1$. So we can use (13) to find $P(\theta)$ and draw all sorts of interesting conclusions. We will use Matlab's built-in ODE solver `ode45` to do this job.

Summary. Using the crank angle θ as a parameter, the interval $-\pi < \theta < \pi$ captures all four thermodynamic states in Figure 1. We have $\theta = -\pi$ in state 1, $\theta \approx 0$ for states 2–3 (often a poor approximation), and $\theta = \pi$ in state 4. We get $V(\theta)$ from geometry (line (9)), $Q(\theta)$ by inventing something plausible (line (11)), and $P(\theta)$ by solving a differential equation (line (13)). Once all three functions are known, we can calculate the work done by the piston as

$$W = \int P\,dV = \int_{\theta=-\pi}^{\pi} P(\theta) V'(\theta)\,d\theta \tag{14}$$

and deduce the cycle's thermodynamic efficiency

$$\eta = \frac{W}{\overline{Q}}. \tag{15}$$

IDEA 3: *Global Variables in Matlab*

Here's some of what Matlab has to say in response to "`doc global`":

> "`global X Y Z` defines `X`, `Y`, and `Z` as global in scope.
>
> Ordinarily, each MATLAB function, defined by an M-file, has its own local variables, which are separate from those of other functions, and from those of the base workspace. However, if several functions, and possibly the base workspace, all declare a particular name as global, they all share a single copy of that variable. Any assignment to that variable, in any function, is available to all the functions declaring it global."

In studying an engine prototype, the physical parameters of the engine are needed in every script and function we create. To make it easy to apply the same tools to different configurations and scenarios, it's nice to have just one reference copy of each parameter, and to share it with all functions that need it. (The alternative would require checking each occurrence of every parameter in every script and every function, every time you wanted to adjust one!) Global variables are good for this.

Global variables have disadvantages, too: if some function changes the value of a global variable, all the other functions that use it will be affected. If the change was unwanted, it can be very hard to figure out which one of all the functions currently active contains the error. Successful use of global variables requires strict limits on where they are allowed to change, and clear identifiers discouraging accidental overwriting. To help with this, it's common to identify global variables as special and draw attention to them somehow. This week's plan is to put the prefix `global_` on each one. Then we agree to take special care whenever we change the value of a global variable: we will use a dedicated script called to initialize all the global variables, and require almost all other scripts and functions to only read, never overwrite, global values. The scripts supplied with the lab follow this practice.

Notice that any script that wants to use a global value must declare its intent to use the variable in the global storage region instead of creating a new local variable with the same name. But to improve the readability of code, it can be nice to make local copies of the global values with simpler names. The given files `vdv.m` and `qdq.m` illustrate this practice.

IDEA 4: *Multi-Pane Plots in Matlab*

Matlab's `subplot` command provides an easy way to generate multi-pane plots whose horizontal axes line up precisly. To set up a window with two plotting zones, one above the other, and aim subsequent instructions at the top zone, say

```
subplot(2,1,1);
```
(16)

All normal plot commands can be used, and they all apply to the top sketch in the two-pane figure. To switch to drawing in the bottom zone, give the command

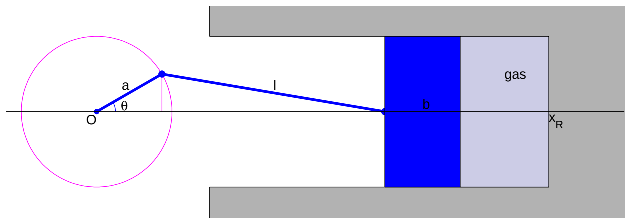
```
subplot(2,1,2);
```
(17)

You can change the focus between zones whenever you like by re-entering one of (16) or (17).

This is all you need to know about `subplot` to complete this lab, but there is more to learn. The online help ("`doc subplot`") provides more information about this versatile command.

Appendix: Piston Geometry

Consider the piston/crank assembly in Figure 2 of the main writeup. The crank pivots at the origin.



For the angle θ shown here, the x-coordinate of the piston's face is

$$x(\theta) = a\cos\theta + \sqrt{\ell^2 - a^2\sin^2\theta} + b.$$

The extreme positions of the piston's active face are clearly

$$\begin{aligned} x_{\mathrm{MAX}} &= x(0) = a + \ell + b, \\ x_{\mathrm{MIN}} &= x(\pi) = -a + \ell + b. \end{aligned}$$

Here a is the length of the crank arm, and the piston's "stroke" is $2a$. If A denotes the area of the piston face, then the volume of the region the piston face moves through in each cycle, called the "displacement", is $V_d = 2aA$. Suppose the top end of the piston chamber is at location x_R. Then the extreme volumes inside the chamber will be

$$\begin{aligned} V_0 \stackrel{\mathrm{def}}{=} V_{\mathrm{MIN}} &= A(x_R - x_{\mathrm{MAX}}) = A(x_R - b - a - \ell), \\ V_{\mathrm{MAX}} &= A(x_R - x_{\mathrm{MIN}}) = A(x_R - b + a - \ell). \end{aligned} \tag{$*$}$$

The standard name for V_{MIN} is the "clearance volume"; the standard symbol is V_0. Hence

$$V_{\mathrm{MAX}} = V_0 + V_d.$$

Piston experts work with two dimensionless ratios:

$$R = \frac{\ell}{a}, \qquad r = \frac{V_{\mathrm{MAX}}}{V_{\mathrm{MIN}}} = \frac{V_0 + V_d}{V_0}.$$

Note that $r - 1 = V_d/V_0$. The volume ratio between enclosed volume and piston displacement is

$$\begin{aligned} \frac{V(\theta)}{V_d} &= \frac{A(x_R - x(\theta))}{A(2a)} = \frac{1}{2a}\left[x_R - b - a\cos\theta - \sqrt{\ell^2 - a^2\sin^2\theta}\right] \\ &= \frac{1}{2}\left[\frac{x_R - b}{a} - \cos\theta - \sqrt{R^2 - \sin^2\theta}\right]. \end{aligned}$$

Solving for $x_R - b$ in $(*)$ gives

$$x_R - b = \frac{V_0}{A} + a + \ell = 2a\frac{V_0}{V_d} + a(1 + R).$$

Back-substitution yields

$$\frac{V(\theta)}{V_d} = \frac{1}{2}\left[2\frac{V_0}{V_d} + 1 + R - \cos\theta - \sqrt{R^2 - \sin^2\theta}\right] = \frac{1}{r-1} + \frac{1}{2}\left[1 + R - \cos\theta - \sqrt{R^2 - \sin^2\theta}\right].$$

In the main lab writeup, we write R_1 instead of R for the length ratio, to avoid a notational collision with the ideal gas constant.